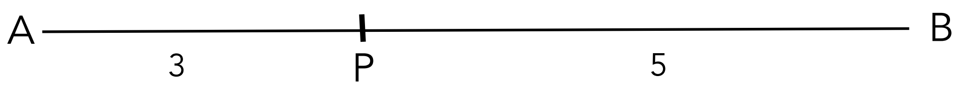
Centroid

Lemma 1.

There is only one point P which divides a line AB in a given ratio.

Proof.

Let P be a point dividing the line AB in the ratio 3 : 5 (as an example).



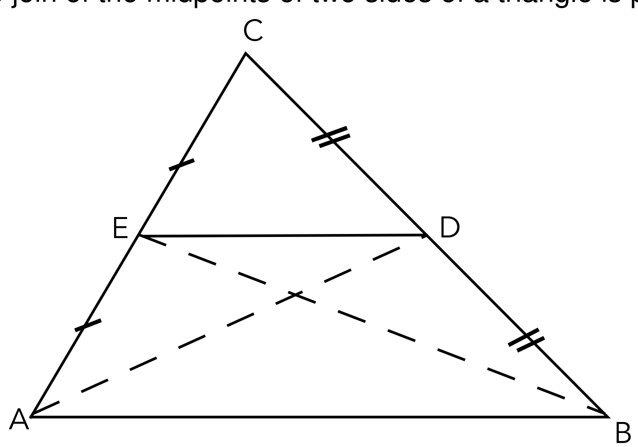
Divide AB into 8 (3 + 5) equal units.

We require that $AP = \frac{3}{8}AB$.

Measure, or 'lay down', starting from A, 3 units to construct the unique point P. [Note we are here converting lengths to numbers, or assuming copies of lengths, or 'superposition' of segments.]

Lemma 2.

The join of the midpoints of two sides of a triangle is parallel to the third side.



Proof.

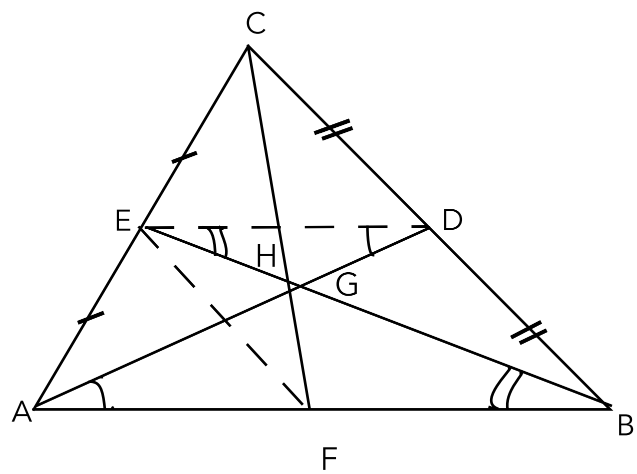
Let D, E be midpoints of BC, AC respectively.
area $\triangle DCE =$ area $\triangle DBE$ (same base and height)
area $\triangle DCE =$ area $\triangle DAE$ (same base and height)
So area $\triangle DBE =$ area $\triangle DAE$.
But $\triangle$s DBE, DAE have common base ED, so their heights are equal and $ED//AB$. $\square$

(Continued)

Theorem.

The medians of any triangle are concurrent.

Proof.



Let D, E, F be the midpoints of BC, CA, AB respectively. By Lemma 2, $ED//AB$, so $\triangle$s EDC, ABC are similar with sides in the same ratio as $CE : CA = 1 : 2$.

Let AD meet BE at G. (The median CF has been deliberately drawn so as not to pass through G.)

Then $\triangle$s GDE, GAB are similar, so $AG = 2GD, BG = 2GE$. Thus G divides AD, BE in ratio $2 : 1$.

Now suppose CF meets BE at H. Join EF. By a similar argument with $\triangle$s HEF, HBC, $BH = 2HE$, so H divides BE in ratio $2 : 1$, and by Lemma 1, H coincides with G. Thus the medians are concurrent at G.

$\square$