29th October   Short Story 1:   Patterns

Starter: To help get in the right mood, orient our thoughts, and accommodate late-comers.

`Automated sequence of images called 'Patterns and Symmetry'.`

Spoken words:

Welcome to everyone – it's good to see so many here. No idea there was so much hidden interest in mathematics in Kenilworth! My plan here in this short chat is not to teach anything (you might learn, or unlearn, some things) but rather to promote a *positive perspective* on mathematics. My hope is that you go away thinking there is more to mathematics than 'getting your sums right' (or wrong). Inevitably this can be the dominant memory of mathematics from schooldays. Whether maths was a favourite for you, or the opposite, the maths we shall look at this evening is (probably) different from anything you may have met with at school. If you like the idea, it's a new start.

We're going to be looking at patterns [slide 1] and as you saw from the 'starter' slides, and from the leaves on this slide, symmetry is a common source of pattern in nature and art. We're going to see how symmetry appears in maths. Patterns are very important in mathematics. Here's a quotation from a book – famous when I was a student – written by G.H. Hardy. [`Slide 2`]

We read the quote largely silently, protestations audible from Victoria at the words 'commonplace' and 'unimportant' [!] Of course, in the light of many of the art images we have just seen, Hardy is just wrong here … But he is spot on when saying the material of mathematical patterns is *ideas,* and the need for patterns to be *beautiful*.

[Slide 3, para1] Our patterns will be made by taking a collection of similar things and combining them, two at a time, to make new things. E.g. with whole numbers we might combine by addition, so 3 + 4 = 7. We can sometimes make a table of the combinations.

I'm lucky that for the sake of my work I go to Prague quite often. One night in Prague, a while back, I could not sleep and heard a clock striking one. 'Oh dear', I thought, '1 o'clock and I'm not feeling at all sleepy'. About 10 minutes or so later I heard the clock strike two. 'Time goes fast in Prague … or perhaps I did fall asleep?', not much later the clock is striking three! You have guessed what was happening. The clock was striking quarter-hours, not hours. [Slide 3, para2]
It's the quarter-hours that are our objects here, not the chimes, and 'followed by' is how we combine them. Leon* will fill in the table on a flipchart. [He first put 1, 2, 3 on the dark grey top row; and 1, 2, 3 on the dark left column. He spoke his thought processes: "on row2, column1, 2 quarter-hours followed by 1 more, makes 3; on row 3, 3q-hs followed by 2, would make 5, but after striking the hour (after 4 q-hs) we 'reset' the number of q-hs to 0, so the answer becomes 1. This means we shall need a '0' in top row and left column." He filled in the rest of the first table on slide 4.] By the 'table' in this context we mean the 4x4 array of light-coloured squares. You might think of the symbols in the dark squares as 'inputs' (along the top, and the left-hand column), with the 4x4 table as the 'outputs'. [*Leon is my invaluable, young assistant in these talks.]

For a more abstract example (slide 3, para 3) consider the fourth roots of 1 being combined by multiplication. Remember a 'fourth root' of a number N is another number which, when multiplied by itself four times makes N ($3^4 = 81$), so 3 is a fourth root of 81 (so is -3). 'Pretend' there is a square root of -1, which we shall call i (imaginary). Then $i^2 = -1$, and so also $i^4 = 1$, and so ±1 and $\pm$I are our objects (our fourth roots). The table can then be completed as shown in slide 4. [Remember a 'negative times a negative makes a positive'.]

The *ideas* behind each table on slide 4 (quarter-hours and fourth roots of 1) seem very different but the patterns made in the tables are exactly the same.

Turning now to symmetry patterns. By a 'symmetry' of an object we mean a *movement* that brings an object into 'coincidence' with itself. So after the movement the object occupies the same space as before (but perhaps a different orientation).

Now take a cuboid (shoebox, matchbox etc) and label the 8 corners clearly. There is a long axis (X), a width axis (Y) and a vertical axis or 'thickness' (Z). Think of these axes going through the centres of the faces of the box. Our objects this time will be half-turns (180°) around each axis. Denote these half turns by the letter of the axis (X, Y, Z). [See slide 5.] The operation to combine them is again 'followed by'. Obviously two half-turns around the same axis brings the box back to how it started (no change, we denote by I for 'identity'). Surprisingly X followed by Y brings the box into the exact same orientation as Z [try it!], and that X followed by Z is the same as Y. [Leon here demonstrated these very well with his labelled shoebox.]
The second and third rows can be completed in a similar way. The finished table is on slide 5.

Another example on slide 5 is the numbers 1, 3, 5, 7 with a special rule of combination: multiply as normal but then divide by 8 and use the *remainder* as the result. See that all the possibilities produce the second table on slide 5. Again the objects and operation are quite different from the cuboid example, but the patterns of the tables are identical.

The patterns we are seeing can be thought of at different levels. The 'top' level is the perceptual level of the the quarter-hours, or the shoebox with its labelled corners. The tables we made of the various combinations make more abstract patterns, a 'deeper' level. The sources can be different with the underlying pattern the same (both slides 4 and 5 are examples). Then the pattern in slide 5 is different from the pattern in slide 4. But the table *properties* in both slides 4 and 5 are the same and are at an even 'deeper' level – all four tables are examples of a *group structure* in mathematics (see slide 6).

The *order* of a group is the number of objects in it. As the order of a group gets bigger there are more and more possibilities for different patterns. The symmetries of an equilateral triangle has order 6: three rotations (in the plane of the triangle, we include here the 360°, or identity, rotation) and three 'flips' about an axis so that exactly two vertices swap position. Another way to get these flips is always to flip about the same axis but to rotate the triangle first – this is what happens on slide 7 - where 'fr' means rotate 120° then flip around vertical axis, fr^2 means rotate 240° then flip around vertical axis. [That's not a misprint, we usually read these expressions like fr^2 'backwards'!] We briefly looked at the website[1] on slide 7 which can be used to help construct your own table for the six symmetries of this triangle. Here it is the movements (a flip, or rotation, or a combination of these) that are the objects, and 'followed by' (*) is the operation on them. You should find that for some objects a and b, a*b is not equal to b*a. For the groups of order 4 such combinations were always equal.

The size of the 'symmetry group' of an object is a kind of measure of how much symmetry the object has. For example, the symmetry group of a cube is of order 24. Read about it on the web.

If you have questions about any of the above, or ideas for improving it, please let me know by email at russ.steve@gmail.com.

We have been looking at the beginning ideas of group theory, it is today a lively and vigorous part of pure mathematics (slide 9) with applications in many different fields of science. It began over 200 years ago (slide 8) and now forms part of most higher level courses in mathematics. Some conclusions are summarised on slide 10, and some sources to follow up on are listed on slide 11 (there are very many others). The book by Ronan is a nice easy read and a survey of an extraordinary mathematical challenge – which has largely been achieved. Many others have not (yet).

The talk led into some challenging and interesting questions and discussion which I have tried to summarise in another file linked from this webpage. I welcome any further comment or questions on these.

Steve Russ

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[1] Also see https://jseden.dcs.warwick.ac.uk/construit/ to explore this environment.